Sao Luis-MA, March 1st, 2025

Dear Editor,

We thank the reviewers for the careful analysis and their comments to the paper SUBMIT2IJMS-D-25-00022, which have allowed us to improve the manuscript.

Changes in the text are highlighted in red to facilitate their location in the manuscript. Also, we transcribe below the reviewers’ comments and present, in blue, our responses.

Associate Editor: The presentation and also depth of the analysis need to be improved. The points below are not by any means exhaustive but should help to enhance this submission.

1. Writing needs improvement. Each section should open with a paragraph or two explaining what to expect before going to subsections. Manuscripts should be written in the third person, passive voice.

We greatly appreciate the suggestion of the Associate Editor. Following the comments of the reviewers, the paper has been reformulated, such that the suggested paragraphs were inserted and the paper written in passive voice.

2. Much more thinking needs to be done how the material should be presented. The rudimentary material (e.g. basic mathematical modelling, technical drawing looking figures, repetitive graphs, etc) should be moved to an appendix or removed; similar figures should be clustered in so-called panel figures.

We thank the Associate Editor for the comment. The equations regarding the numerical methods were transferred to appendices,since the novelty of our work is not related to the derivation of the numerical approaches, but to the application of them in our proposed problem. The figures were also revised, in such a way that repetitive schematics were clustered in panels or removed. Please, see the revised version of the paper, see Figure 1, for example.

3. Mathematical formulations need to be much more compact. An equation is a part of a sentence, so punctuation marks like comma and full stop apply.

We agree with the Associate Editor. We revised the equations of the manuscript and added punctuation remarks where needed. Please, check the revised version.

4. Broaden and update the literature review to better connect to the current effort in the field in the context of mechanical sciences; for papers like this one we expect no less than 80 journal papers including 25 recent ones (2023/24->) to be critically discussed. Do not cite textbooks or manuals.

We thank the Associate Editor for this important comment. We extended the literature review in our paper, providing recent and relevant research related to the work. We also discussed along the introduction and simulated examples sections, notable techniques on how the literature is currently approaching the problem stated in our research. Please, see the revised version of the paper.

5. The choice and fidelity of keywords is deficient. The first three keywords should generic and the remaining three paper specific.

We thank the Associate Editor for the suggestion. The keywords were reformulated to better describe the work. Please, see the revised version of the paper.

6. Novelty needs to be explicit in highlights and abstract.

We thank the Associate Editor for the comment. The highlights and the abstract were revised in order to provide clearly the main contributions and novelty of our work.

7. Highlights should be punchy short bullet points. Focus what and why is novel, not just state it is novel. 85 characters is the length limit. Do not use acronyms in highlights (and Conclusions).

We thank the Associate Editor for this important suggestion. The highlights of the paper were revised following the suggestions. Additionally, the acronyms were also removed from the conclusions section of the paper. Please, see attached the revised version with the corrections.

8. Graphical abstract has way too much information and is not of proper format, it should be landscape. Get familiar what constitute a good graphical abstract and produce one; for sure simplify it as we typically see the GA in the email announcements and we have only a second or less to get interested or not.

We thank the Associate Editor for the suggestions. The graphical abstract was revised accordingly, and it is presented now in landscape. The amount of information on the graphical abstract was adjusted to provide a smoother appearance to the reader. Please, see the revised graphical abstract attached.

9. Figures require much more fidelity and density of information as currently some are good for an internal report only. Graphs need to be improved, e.g. box graphs, remove grids and minor ticks, move ticks inside, limit number of ticks to 4-6 per axis, line thicknesses should differentiate between curves and axes, fonts should be of the same type and size for all figures etc. Panel labels should be in regular font and in round brackets, e.g. (a), located just above the top left corner of figure panels. I am attaching example figures to follow.

We appreciate the comment of the Associate Editor. The figures in the paper were revised and corrected according to the given specifications. Please see the revised version of the paper.

10. Expand figure captions so figures are almost self-explanatory. An example is given in one of the attachments.

We thank the Associate Editor for the suggestions. The figure captions were revised in order to provide clear information about the content of themselves. We greatly appreciate the examples given, and they were used as references to improve the quality of our work. Please, see the revised version of the paper.

Reviewer #1: The paper needs in revision:

1. There are several more capable methods for analyzing dispersion of surface waves in beams, two of them should be mentioned, Cauchy formalism and Stroh formalism, both can be used for studying symmetric and asymmetric modes, and both are free from simplified Bernulli - Euler hypothesis. The authors should at least mention alternative approaches.

We thank Reviewer #1 for the interesting suggestion. These two formalisms were cited in our work. Please see section 2: Graded elastic metamaterial beam modelling*.* Furthermore, the Euler-Bernoulli theory for flexural vibration of beams was chosen intentionally in our study, since despite the simplifications of this hypothesis, this model is widely used to represent flexural waves in 1-D engineering structures, for example, see [17, 79].

[17] Y. Xiao, J. Wen, X. Wen, Broadband locally resonant beams containing multiple periodic arrays of attached resonators, Physics Letters A 376 (16) (2012) 1384–1390. doi:10.1016/j.physleta.2012.02.059.

[79] E. J. P. Miranda Jr., J. M. C. Dos Santos, Flexural wave band gaps in phononic crystal Euler-Bernoulli beams using wave finite element and plane wave expansion methods, Materials Research 20 (Suppl.2) (2017) 729–742. doi:10.1590/1980-5373-MR-2016-0877.

2. What is the main reason in performing computations with different FE approaches. It looks strange in comparing the spectral FE with FE at the same number of mesh element; this also refers to other FE approaches. Give a more detailed explanation of the mesh size, element type, spectral element order, etc.

We thank Reviewer #1 for the important comment. In our work, we chose to perform a multi-approach (plane wave expansion, extended plane wave expansion, wave finite element, and wave spectral element methods) evaluation of a graded locally resonant metamaterial beam, which, to the best of our knowledge, was not performed before in literature. The wave finite element is an approach based on the finite element method, in the sense that it uses its dynamic stiffness matrix, to model a slice (substructure) of a periodic structure. This wave-based model does not require the discretization of the domain into small elements, but a partitioning into convex subdomains and is applied to describe the dynamics of periodic structures by computing frequencies and its wave propagation modes, see [80]. On the other hand, the wave spectral element method is also a wave-based approach that is based on the dynamic stiffness matrix obtained by the spectral element formulation, which by definition is an analytical solution for the motion of the model. Regarding the comparison between these two methods, the wave spectral element solution is analytical, therefore, more suitable for high frequency analysis, which can be easily extended using the same formulation presented in our work. Furthermore, these methods are less computational expensive when compared to the finite element approach, for example. In both cases, the 1-D Euler-Bernoulli beam element was considered, therefore with two degrees of freedom in each element node. The spectral element discretization was considered in such a way that each resonator in a unit cell coincides with the spectral element node, i.e., a discontinuity in the model. The wave finite element discretization used a partitioning of the unit cell in 20 substructures (slices). The dynamic stiffness matrix for each substructure was, therefore, of order 36. This comment was also included in the revised version of paper, please see section 3.1 Simulated examples.

[80] Mencik, J.-M. On the low- and mid-frequency forced response of elastic structures using wave finite elements with one-dimensional propagation. Computers and Structures. 2010; 88:674-689.

3. Give a plot for the "normal" dispersion of flexural waves in a beam without resonators, otherwise the presented dispersion plots are useless.

We agree with Reviewer #1 and follow the suggestion. The dispersion diagram for a bare beam, i.e., without attached resonators was included in the simulated examples. Please, see subsection 3.1.1, Figure 2, of the revised version of the paper.

4. Considering wave propagation analysis, give explanation on the time-integration schemes, were they energy preserving (like Lax - Wendroff), did authors use a single time integration schemes for all spatial FE discretization?

All beam modeling was done in the frequency domain to generate a harmonic solution. From this point on, the energy equations for all models analyzed in this paper were used to obtain the mass and stiffness matrices of a finite element. There are several methods that can be used, and in our work, the finite element displacement method was used. In this way, it is possible to build a structure repeating a unit cell periodically, thus, the wave constant can be used to predict the behavior of the entire structure [86]. The WFE formalism uses numerical wave modes as expansion bases for describing the kinematic variables of these structures, that is the displacements and external/internal forces. The wave modes are numerically computed using the finite element (FE) model of a typical substructure [80].

[80] Mencik, J.-M. On the low- and mid-frequency forced response of elastic structures using wave finite elements with one-dimensional propagation. Computers and Structures. 2010; 88:674-689.

[86] D. J. Mead, A general theory of harmonic wave propagation in linear periodic systems with multiple coupling, Journal of Sound and Vibration 27 (2) (1973) 235–260. doi:10.1016/0022-460X(73)90064-3.

5. It is known that summation of series in the right sides of Eqs. (6) - (8) leads to the ill-posed problem in L2 topology. How did authors manage to obtain their numerical results, what kind of regularization they used, how the non-physical oscillations were removed, etc.

Thanks for this important comment and question. The Eqs. (6) and (7) (moved to Appendix A, as suggested by the Associate Editor, now become A.4 and A.5), are the main idea associated with the plane wave expansion (PWE) approach [17, 41, 45, 78], i.e., the use of Fourier series expansion in spatial domain and Floquet-Bloch theorem. We highlight that no regularization was used for the PWE method similar as the majority of studies which used this approach (see, for example, [17, 41, 45, 78]), since the number of plane waves used in this article (i.e., 3) is suitable for the Fourier convergence in cases of 1-D and 2-D mechanical metamaterials with attached spring-mass resonators [17, 41, 45, 78]. We added a comment about this issue in this article to clarify. Please, see Appendix A. Finally, the Eq. (8) (Appendix A, A.6) is only the Fourier series expansion in spatial domain of the sum of Dirac delta function calculated similar as previous studies [17, 41, 45, 78].

[17] Xiao, Y.; Wen, J.; Wen, X. Broadband locally resonant beams containing multiple periodic arrays of attached resonators. Physics Letters A. 2012; 376:1384-1390."

[41] Miranda Jr., E.J.P; Dos Santos, J.M.C. Flexural wave band gaps in multi-resonator elastic metamaterial Timoshenko beams. Wave Motion. 2019; 91:102391.

[45] Dal Poggetto, V.F.; Miranda Jr., E.J.P.; Dos Santos, J.M.C.; Pugno, N.M. Wave attenuation in viscoelastic hierarchical plates. International Journal of Mechanical Sciences. 2022; 236:107763.

[78] Miranda Jr., E.J.P.; Rodrigues, S.F.; Aranas Jr., C.; Dos Santos, J.M.C. Plane wave expansion and extended plane wave expansion formulations for Mindlin-Reissner elastic metamaterial thick plates. Journal of Mathematical Analysis and Applications. 2022; 505:125503.

Reviewer #2: This work numerically investigates the bandgap and wave attenuation characteristics of locally resonant metamaterial beams with periodically attached spring-mass. The coupling effect of the bandgaps and the widening of the wave attenuation domain can be achieved by graded design. The research content of this paper is rich and the framework is well organized. I suggest that it be published after minor revisions. Some comments are given below:

We appreciate this kind comment of Reviewer #2.

1. In the first section, the author needs to introduce the engineering application scenario of the beam with periodic spring-mass graded resonators.

We agree with Reviewer #2 and we follow the suggestion. Several engineering applications of locally resonant metamaterial beams with graded resonators have been added to section 1, introduction.

“Within an engineering framework, metamaterial beams with periodic graded resonators have been proposed for energy harvesting [61-63], energy trapping, confinement, and amplification [64, 65], mode conversion [66], micro-electromechanical systems (MEMS) [67, 68], low frequency broadband sound absorption, and waveguides [69, 70].”

To the best of our knowledge, the large scale insertion of metamaterials in industry has been hindered for challenges, namely, reliability of models regarding fatigue life, robustness of additive manufacturing processes, and system integration. These challenges provide new opportunities for future research and pave the way for further exploration in the next generation of advanced materials.

2. The Brillouin region of the graded periodic structure is different from the homogeneous one, which should be drawn in the article.

We thank Reviewer #2 for suggesting this improvement. An illustration of the reciprocal lattice for a 1-D periodic system was added in section 2, Figure 1(b). For homogeneous structure, i.e., bare beam without resonators, the periodicity is not defined, since there are no discontinuities in the system. Therefore, the length of the unit cell, or the lattice constant, , can be chosen arbitrarily.

3. Will graded design substantially increase the overall mass of metamaterials? Large added mass should be avoided as much as possible in engineering.

We thank Reviewer #2 for the important question and comment. In our study, the overall mass of the resonators is given by the parameter , Eq. 4. That means that the sum of mass of the mass of all resonators is constant, no matter which kind of progression was considered in the mass distribution. Additionally, in subsection 3.3, we performed a parametric analysis on the effect of varying the parameter with respect to the smallest part of the imaginary component of the normalized wave number, see Figure 13. For the case of simultaneous gradation in resonator’s position and mass, the additional mass constants, , , and , Eqs. 8-10, regarding AP, GP, and QP respectively, were also evaluated according to a parametric variation with respect to the smallest part of the imaginary component of the normalized wave number, see Figure 14. For practical engineering applications, high resonator-structure mass ratios are avoided by design, justifying the performed parametric analysis carried out in our study.

4. The conclusion is too cumbersome and it is recommended to summarize in bullet point mode.

We thank Reviewer #2 for this important suggestion. The conclusion section of the paper was written in such a way to recap the main idea and contribution of the work, since many cases, numerical approaches, and results were presented. After this, the conclusion was reformulated in summarized bullet point topics.

5. It seems that the authors have ignored the recent research progress related to this paper, which implements the bandgap coupling effect to broaden the vibration attenuation domain: <https://doi.org/10.1016/j.apm.2023.09.030>.

We appreciate the comment of Reviewer #2. We have now included the mentioned reference regarding the coupling effect between Bragg scattering and local resonance. Additional literature concerning this phenomenon has also been added to our discussion. Please, see subsection 3.1.2 Geometric progression:

“The LR and Bragg type bandgap coupling is a phenomenon reported in literature as a prominent manner of enhancing a metamaterial performance, i.e., broadening vibration attenuation ranges [81-84]. Moreover, the advantage of tuning the resonator resonance close to the Bragg scattering frequency has also been reported for analogous electromechanical systems [85].”

[81] B. Yuan, V. F. Humphrey, J. Wen, X. Wen, On the coupling of resonance and Bragg scattering effects in three-dimensional locally resonant sonic materials, Ultrasonics 53 (2013) 1332–1343. doi:10.1016/j.ultras.2013.03.019.

[82] M. Chen, D. Meng, H. Zhang, H. Jiang, Y. Wang, Resonance-coupling effect on broad band gap formation in locally resonant sonic metamaterials, Wave Motion 63 (2016) 111–119. doi:10.1016/j.wavemoti.2016.02.003.

[83] A. O. Krushynska, M. Miniaci, F. Bosia, N. M. Pugno, Coupling local resonance with Bragg band gaps in single-phase mechanical metamaterials, Extreme Mechanics Letters 12 (2017) 30–36. doi:10.1016/j.eml.2016.10.004.

[84] Z. Deng, B. Zhang, K. Zhang, L. Peng, P. Liu, Q. Sun, F. Pang, The coupled band gap of longitudinal wave in metamaterial-based double-rod containing resonators, Journal of Vibration and Control (2024). doi:10.1177/10775463241276706.

[85] Y. Sun, Q. Han, T. Jiang, C. Li, Coupled bandgap properties and wave attenuation in the piezoelectric metamaterial beam on periodic elastic foundation, Applied Mathematical Modelling 125 (2024) 293–310. doi:10.1016/j.apm.2023.09.030.